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LAMINAR VISCOUS FLOW THROUGH REGULAR ARRAYS OF PARALLEL SOLID CYLINDERS

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Abstract—Solutions are found for the Stokes equations of motion for a viscous fluid flowing either parallel or perpendicular to the axes of cylinders in square, rectangular, triangular and hexagonal arrays. This is done by matching a solution outside one cylinder to a sum of solutions with equal singularities inside every cylinder of an infinite array. Some of the terms in the solution are indeterminate but these indeterminacies are resolved. The resulting solutions are several terms of a power series in the density.

High density approximations are found for the longitudinal case when the cylinders overlap.

For low densities the mean velocity for transverse flow is found to be independent of orientation of the array and is half the mean velocity for parallel flow in the same pressure field to several orders of magnitude of the volume concentration of cylinders.

1. INTRODUCTION

The theoretical study of the flow of a viscous fluid past a regular array of cylinders is part of the wider study of the relative motion of a mixture of a fluid and solid bodies. The parallel flow solutions are idealised solutions for the flow through cigarette filters, plant stems and around pipes in heat exchange tanks. The transverse solutions are applicable to transverse fibrous filters used for cleaning liquids and gases and regulating their flow. Both types of solutions are also applicable to the settling of suspensions of long thin particles. The high density solutions show some of the properties of flow through porous substances.

This study has also some mathematical interest in that it involves the validation of some indeterminate factors and the use of analytic continuation around singularities.

The aim of this paper is to find solutions for several different configurations of cylinders. These solutions include both low and high density approximations which match up reasonably well at intermediate densities.

The parallel flow problem was first solved by Emersleben (1925) for a square array. His solution which was based on complex zeta functions is valid only for small densities.

Happel (1959) employed a very simple approximation called a free-surface model. In this model, the liquid associated with each cylinder is lumped into a concentric cylinder with zero drag on the surface. He derived a formula for the Kozeny constant, which is equivalent to a drag force F per unit length of the cylinder, given by

$$F = \frac{4\pi\mu U}{\ln(1/\epsilon) - 1.5 + 2\epsilon - \frac{1}{2}\epsilon^2},$$

where U is the average speed of the fluid and μ is the fluid viscosity. This formula is moderately accurate for small values of the cylinder density ϵ for a triangular packing.

†Iqbal Tahir shared in the work on square arrays.

Sparrow & Loeffler (1959) solved the parallel flow problem for a square and an equilateral triangular array by using a truncated variable-separable solution which was exact on three boundaries of a typical element of a cell of the array and was collocated or fitted at a set of points on the fourth boundary of the element. Their results were graphed and agreed well with the formulae of Emersleben and Happel and extended them up to touching densities.

Banerjee & Hadaller (1973) minimised $\int((\text{grad}\,\phi)^2 - \phi)\,dV$, where ϕ is the velocity potential, with eighth order polynomials and graphed their solutions for the triangular lattice for $b/2a$ from 1.01 to 4 where b is the distance between the centres of adjacent cylinders and a is the cylinder radius.

For transverse flow the equations are more complicated.

Tamada & Fujikawa (1957) studied, on the basis of Oseen's equations of motion, the steady two dimensional motion of a viscous fluid passing perpendicularly through an infinite row of cylinders and calculated the drag acting on a cylinder in the row.

Hasimoto (1959) used Fourier series to calculate the drag force F_1 per unit length for flow perpendicular to a square grid of cylinders, obtaining

$$F_1 = \frac{4\pi\mu U_1}{\ln(b/a) - 1.3105},$$

where U_1 is the mean velocity of the fluid.

He also stated that by using elliptic functions, he had obtained another term $\pi a^2/b^2 + O(a^4/b^4)$ in the denominator.

Happel (1959) used his free surface model to calculate the drag force

$$F = \frac{4\pi\mu U}{\ln(l/a) - \frac{1}{2}\ln\pi - \frac{1}{2}}$$

where l is the distance between the centres of adjacent cylinders.

As the shear force is non-zero on the upstream and downstream faces of a cell, a better assumption is that of Kuwabara (1959) who started with a model identical with that of Happel's but with a vorticity free boundary condition and found the drag force per unit length, equivalent to

$$F = \frac{4\pi\mu U}{\ln(l/a)\,\frac{1}{2}\ln\pi - 3/4 + \pi a^2/l^2 + O(a^4/l^4)}$$

Kirsch & Fuchs (1967) tested Happel's and Kuwabara's formulae using a regular triangular lattice and found good agreement especially with the latter.

Spielman & Goren (1968) used the dipole approximation to the pressure field and represented the effect of other cylinders as an additional body force which they calculated to comparable accuracy.

Gordon (1978) obtained Kuwabara's solution for the stream function and the vorticity by using a finite difference iterative procedure.

Sangani & Acrivos (1982a) used a collocation of cylindrical biharmonics on the outer boundary for 10 different values of the density c for both square and triangular arrays. For high densities they obtained the lubrication type approximations for narrow gaps.

They (1982b) extended Hasimoto's method to obtain

$$F = \frac{4\pi\mu U}{\frac{1}{2}\ln(1/c) - 0.738 + c - 0.887c^2 + 2.038c^3 + O(c^4)}$$

for a dilute square aray and

$$F = \frac{4\pi\mu U}{\frac{1}{2}\ln(1/c) - 0.745 + c - \frac{1}{4}c^2 + O(c^4)}$$

for a triangular array.

There is a considerable literature of experimental results reviewed by Davies (1973). The above theoretical results for regular lattices do not agree particularly well with the experimental results of Sullivan (1942). To overcome this difficulty Yu & Soong (1975) proposed a random cell model choosing a spectrum of cell sizes to fit the experimental results.

Other similar transport problems around cylinders occur in electrical conduction, heat flow and optics. Work in these areas has been done by Lord Rayleigh (1892), Drummond (1971), Ninham & Sammut (1978) and Perrins *et al.* (1979) using a method of singularities.

This paper adapts the method of singularities to biharmonic equations and produces some rigorous and reasonably accurate formulae for several different arrangements of cylinders, which can be used as a reliable basis for further study of less regular real arrangements of cylinders.

The validity of Lord Rayleigh's method of singularities or images has been questioned by Levine (1966), Jeffrey (1973) and Happel & Brenner (1973) because of what they think are convergence problems for some of the terms.

Lord Rayleigh (1892), Perrins *et al.* (1979) and O'Brien (1979) have, however, described ways to calculate these terms. O'Brien's method which involves a Green's integral around an enclosing rectangle is adapted to the solution of the fluid flow equations in this paper.

2. PARALLEL FLOW THROUGH A SQUARE ARRAY

The description of the square array (figure 1) and the justification for the second solution are given in appendix 1.

The Laurent series for which w, the z component of velocity, has square symmetry and is zero when $r = a$ is

$$w = \frac{P}{4\mu}(a^2 - r^2) + B\ln\left(\frac{r}{a}\right) + \sum_{n=1}^{\infty} B_n\left[\left(\frac{a}{r}\right)^{4n} - \left(\frac{r}{a}\right)^{4n}\right]\cos 4n\theta. \qquad [1]$$

For the multisingularity solution let (r_{pq}, θ_{pq}) be the polar coordinates of a field point $P(r, \theta)$ referred to the centre of another cylinder with centre at (pl, ql) where p, q are

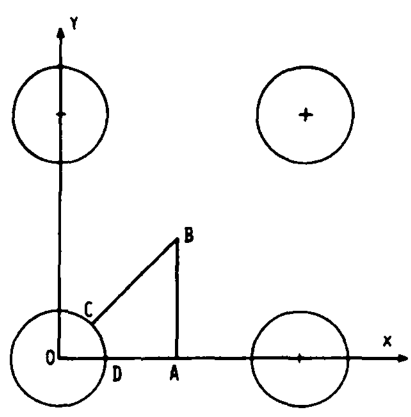

Figure 1. Square array.

integers and l is the distance between the centres of adjacent cylinders. Apart from a constant K, contributed by the outer integral of the Green's solution, the solution with identical singularities at every grid point including the origin is obtained by summing harmonics over a centrally symmetric square region which is then expanded to infinity, and is given by

$$w = K - \frac{Pr^2}{4\mu} + \lim_{I\to\infty} \sum_{p=-I}^{I} \sum_{q=-I}^{I} \left\{ B \ln\left(\frac{r_{pq}}{b_{pq}}\right) + \sum_{n=1}^{\infty} B_n \left(\frac{a}{r_{pq}}\right)^{4n} \cos 4n\theta_{pq} \right\} \qquad [2]$$

where I is an integer.

Matching solutions

Equations [1] and [2] represent the same function in the domain (all θ, $a \leqslant r < l$). Hence

$$Pa^2/4\mu - \sum_{n=1}^{\infty} B_n (r/a)^{4n} \cos 4n\theta = K + \sum_{p,q \neq 0,0}\sum \left\{ B \ln\left(\frac{r_{pq}}{b_{pq}}\right) + \sum_{m=1}^{\infty} B_m \left(\frac{a}{r_{pq}}\right)^{4m} \cos 4m\theta_{pq} \right\}. \qquad [3]$$

Matching may be done using a power series in complex variables as follows.

Let $z = re^{i\theta}$, $z_{pq} = r_{pq} e^{i\theta_{pq}}$ and $d_{pq} = (p + iq)l$, then $z_{pq} = z - d_{pq}$ and [3] may be written as

$$\frac{Pa^2}{4\mu} - \mathscr{R} \sum_{n=1}^{\infty} B_n \left(\frac{z}{a}\right)^{4n} = \mathscr{R}\left[K + \sum_{p,q \neq 0,0}\sum \left\{ B \ln\left(1 - \frac{z}{d_{pq}}\right) + \sum_{m=1}^{\infty} B_m \left(\frac{a}{z - d_{pq}}\right)^{4m} \right\} \right], \qquad [4]$$

where $\mathscr{R}$ is "real part of".

The r.h.s. of [4] may then be expanded as a power series in z and matched term by term to the l.h.s. To do this we define a set of constants

$$P_n = \sum_{p,q \neq 0,0}\sum (l/d_{pq})^n = \sum_{p,q \neq 0,0}\sum (p + iq)^{-n}. \qquad [5]$$

For a square grid symmetric in p and q, P_n is zero if n is not a multiple of 4. Hence the coefficient of z^{4n} in [4] is

$$B_n = \frac{B}{4n} P_{4n} (a/l)^{4n} - \sum_{m=1}^{\infty} B_m \frac{(4n + 4m - 1)!}{(4n)!(4m - 1)!} P_{4n+4m} (a/l)^{4m+4n}, \quad n = 1, 2, \ldots \qquad [6]$$

Equation [6] can be solved step by step in powers of (a/l) to give

$$B_n = B(a/l)^{4n} \left\{ \frac{P_{4n}}{4n} - \frac{(4n+3)!}{4!(4n)!} P_4 P_{4n+4} (a/l)^8 - \frac{(4n+7)!}{8!(4n)!} P_8 P_{4n+8} (a/l)^{16} + \frac{7!(4n+3)!}{3!4!4!(4n)!} P_4 P_8 P_{4n+4} (a/l)^{16} + \cdots \right\}. \qquad [7]$$

Determination of P_{4n}

$$P_{4n} = \sum_{p,q \neq 0,0}\sum (p + iq)^{-4n} = \sum_{p,q \neq 0,0}\sum \frac{\cos 4n(\arctan(q/p))}{(p^2 + q^2)^{2n}}.$$

For large n these series converge rapidly, while for small n we can sum over p to reduce

Table 1. Values of P_n and Q_n for a square grid.

n	P_n	Q_n
2	$-\pi$	
4	3.15121 20021 53895	4.07845 11611 614
8	4.25577 30353 65104	4.51551 54350 320
12	3.93884 9012	3.88073 08453
16	4.01569 5033	4.03154 03146
20	3.99609 6753	3.99219 86989
24	4.00097 6805	4.00195 41008
28	3.99975 5875	3.99951 17843
32	4.00006 1036	4.00012 20739
36	3.99998 4741	3.99996 94826
40	4.00000 3815	4.00000 76294
44	3.99999 9046	3.99999 80926
$4r$	$4+(-1)^r 4^{1-r}+4^{1-2r}+\ldots$	$4+(-1)^r 8.4^{-r}+16^{1-r}+\ldots$

double sums to single sums and obtain

$$P_4 = \frac{\pi^4}{45} + \frac{2\pi^4}{3}\sum_{q=1}^{\infty}\frac{3+2S_q}{S_q^2},$$

$$P_8 = \frac{\pi^8}{4725} + \frac{2\pi^8}{315}\sum_{q=1}^{\infty}\frac{315+420S_q+126S_q^2+4S_q^3}{S_q^4},$$

$$P_{12} = \frac{1382\pi^{12}}{638512875} + \frac{2\pi^{12}}{155925}\sum_{q=1}^{\infty}$$

$$\times\frac{155925+311850S_q+197505S_q^2+42240S_q^3+2046S_q^4+4S_q^5}{S_q^6},$$

etc. where $S_q = \sinh^2 q\pi$. The values of P_{4n} are given in Table 1.

Determination of B

For steady flow of fluid in the channel $ABCD$ of (figure 1), the pressure and shear forces must balance. The thrust due to pressure of unit length of the channel is $P(l^2 - \pi a^2)/8$. The only non-zero shear force is on the face CD. This is

$$\int_0^{\pi/4} \mu(\partial w/\partial r)_{r=a}\, a\, \mathrm{d}\theta = (\mu B - Pa^2/2)\pi/4.$$

Hence, equating these forces for steady flow, we get

$$B = Pl^2/2\pi\mu. \qquad [8]$$

Equations [1], [7] and [8] may now be combined to give the flow velocity

$$w = \frac{P}{4\mu}(a^2 - r^2) + \frac{Pl^2}{2\pi\mu}\left\{\ln\left(\frac{r}{a}\right) + \sum_{n=1}^{\infty}\left[\frac{P_{4n}}{4n} - \frac{(4n+3)!}{4!(4n)!}P_4P_{4n+4}(a/l)^8 + \cdots\right]\right.$$

$$\left.\times\left(\frac{a}{l}\right)^{4n}\left[\left(\frac{a}{r}\right)^{4n} - \left(\frac{r}{a}\right)^{4n}\right]\cos 4n\theta\right\}. \qquad [9]$$

The total flux and drag force

Using [9], the flux Q over a cell of area l^2 around a cylinder is given by

$$Q = 8\int_0^{\pi/4} d\theta \int_a^{l/(2\cos\theta)} w(r,\theta) r\, dr$$

$$= \frac{Pl}{2\pi\mu}\left[\ln\left(\frac{l}{a}\right) + \frac{\pi}{6} - \frac{3}{2} - \frac{1}{2}\ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}P_{4n}}{n(2n+1)(4n+1)4^{n+1}} + \frac{\pi a^2}{l^2}\right.$$

$$- \frac{\pi^2 a^4}{4l^4} + \frac{P_4}{3}\left(\frac{a}{l}\right)^8\left\{-1 + \sum_{n=1}^{\infty}\frac{(-1)^n(4n+3)P_{4n+4}}{4^{n+1}}\right\} + P_8\left(\frac{a}{l}\right)^{16}$$

$$\left.\times\left\{\frac{2}{21} + \sum_{n=1}^{\infty}\frac{(-1)^n 2(4n+7)!P_{4n+8}}{8!(4n+2)!4^n} + \frac{35P_4}{3}\left[1 - \sum_{n=1}^{\infty}\frac{(-1)^n(4n+3)P_{4n+4}}{4^{n+1}}\right]\right\} + O\left(\frac{a}{l}\right)^{24}\right].$$

Three of these series approximate to $3P_4/4$ or $P_8/8$ to 8 digit accuracy. Hence

$$Q = \frac{Pl^4}{2\pi\mu}\left[\ln\left(\frac{l}{a}\right) - 1.310532926 + \pi\left(\frac{a}{l}\right)^2 - \frac{\pi^2}{4}\left(\frac{a}{l}\right)^4 - \frac{P_4^2}{4}\left(\frac{a}{l}\right)^8\right.$$

$$\left. + P_8\left(\frac{35}{4}P_4^2 - \frac{P_8}{8}\right)\left(\frac{a}{l}\right)^{16} + O\left(\left(\frac{a}{l}\right)^{24}\right)\right].$$

If we define the superficial velocity parallel to the cylinders as U = flux per cell/total cell area, the total force per unit length on one cylinder including pressure on its ends as $F = Pl^2$, and the density of cylinders as $\epsilon = \pi a^2/l^2$ and replace the last two calculated terms by a geometric series using the Aitken or Padé transformation, then

$$F = \frac{4\pi\mu U}{\ln\left(\frac{1}{\epsilon}\right) - 1.476335966 + 2\epsilon - \frac{1}{2}\epsilon^2 - 0.0509713\epsilon^4/(1 + 1.51978\epsilon^4)}. \qquad [10]$$

For a high density approximation we assume that the cylinders overlap, or almost overlap, so the walls of the flow channels become the walls of four cylinders with central axes distant $l/\sqrt{2}$ from the central axis of a channel. The channels are isolated if the radius, a, lies between $l/2$ and $l/\sqrt{2}$. The flow is stagnant in the corners and mainly limited by the drag on the parts of the walls nearest to the centre of a channel.

To calculate the flux we use three approximations: (a) two terms of the variable-separable solution of the Stokes flow equations are used, (b) c/a is small where $c = l/\sqrt{2} - a$ is the half gap width, (c) the no-slip wall condition is satisfied only for the part of a wall nearest to the centre of a flow channel.

On one of the four cylinders $x \doteqdot c + y^2/2a$ or $r\cos\theta \doteqdot c + (c^2/2a)\tan^2\theta$ if (y/a) is small. If the flow velocity, w, is zero on this surface then

$$w = \frac{P}{4\mu}\left[c^2 - r^2 + \frac{(a+c)(c^4 - r^4\cos 4\theta)}{(3a-c)2c^2}\right]$$

in the region

$$\left(-\frac{\pi}{4} < \theta < \frac{\pi}{4}\right), \quad \left(0 < r < \frac{c}{\cos\theta}\left(1 + \frac{c}{2a}\tan^2\theta\right)\right).$$

Hence within the above approximations the flux

$$Q = \frac{Pc^4}{\mu}\left(\frac{49}{90} + \frac{487}{1890}\frac{c}{a} + O\left(\frac{c}{a}\right)^2\right)$$

or

$$F \doteqdot \frac{7.35\mu U}{\left[1-\left(\frac{2\epsilon}{\pi}\right)^{1/2}\right]^4\left[1+0.473\left[\left(\frac{\pi}{2\epsilon}\right)^{1/2}-1\right]+\cdots\right]}. \quad [11]$$

This formula may be compared with the formula for a square channel for which a is infinite and the constant 7.35 is replaced by 7.21. Also for a circular channel $c = -a$ so the series $(49/90 - 487/1890 + \cdots)$ should sum to $\pi/8$. Finally, the approximation must fail if $a \ll l/2$ when the channels join up so the flow in the corners is no longer stagnant.

Numerical values from [10] and [11] are listed in Table 5 and compared with values calculated from Sparrow & Loeffler's (1959) (figure 7).

Equation [10] is very accurate for small ϵ and agrees within 2% with Sparrow & Loeffler's values up to $\epsilon = \pi/4$. Equation [11] also agrees with these at densities near $\pi/4$.

3. PARALLEL FLOW THROUGH A TRIANGULAR ARRAY

In this array the solid cylinders are parallel to the z-axis with centres at $[(p + q/2)l, \frac{1}{2}ql\sqrt{3}]$ in the xy-cross section with flow in the z direction. A typical twelfth flow cell is $ABCD$ in (figure 2).

The flow is symmetric about $0A$, $0B$ and AB. The fluid velocity is

$$w = B' \ln\left(\frac{r}{a}\right) + \frac{P}{4\mu}(a^2 - r^2) + \sum_{n=1}^{\infty} B'_n\left[\left(\frac{a}{r}\right)^{6n} - \left(\frac{r}{a}\right)^{6n}\right]\cos 6n\theta.$$

For force balance

$$B' = \sqrt{3}\, Pl^2/4\pi\mu.$$

For matching to the second form of solution we define

$$P'_{6n} = \sum_{p,q \neq 0,0}\sum (p + q/2 + iq\sqrt{3}/2)^{-6n}.$$

The values of P'_{6n} for the triangular grid are given in Table 2.

On matching the two solutions we find that

$$B'_n = B'\left(\frac{a}{l}\right)^{6n}\left[\frac{P'_{6n}}{6n} - \frac{(6n+5)!}{6!(6n)!}P'_6 P'_{6n+6}\left(\frac{a}{l}\right)^{12} + O\left(\frac{a}{l}\right)^{24}\right].$$

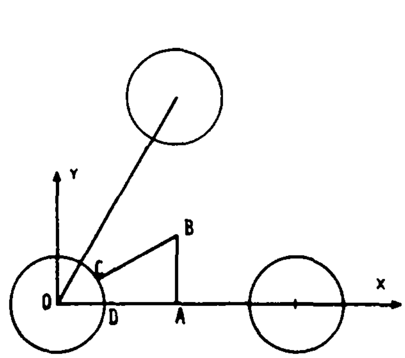

Figure 2. Triangular array.

Table 2. Values of P'_n and Q'_n for a triangular grid.

n	P_n'	Q_n'
2	-3.62759 8728	-
4	0	1.81379 9364
6	5.86303 1693	5.65680 2871
12	6.00963 9972	6.03018 4467
18	5.99971 8358	5.99917 9180
24	6.00001 1648	6.00003 5300
30	5.99999 9587	5.99999 8768
36	6.00000 0016	6.00000 0047
6r	$6+(-1)^r 6/27^r+6/64^r+..$	$6+(-1)^r 18/27^r+24/64^r+..$

The flux through the hexagon of area $\sqrt{3}l^2/2$ surrounding a cylinder is

$$\frac{3Pl^4}{8\pi\mu}\left[\ln\left(\frac{l}{a}\right)-1.3930379468+\frac{2\pi a^2}{\sqrt{3}\,l^2}-\frac{\pi^2a^4}{3l^4}-\frac{P_6'^2}{6}\left(\frac{a}{l}\right)^{12}+O\left(\left(\frac{a}{l}\right)^{24}\right)\right].$$

If the superficial velocity $U=2\times\text{flux}/\sqrt{3}l^2$ and $\epsilon=2\pi a^2/\sqrt{3}l^2$, then the total force on unit length of a cylinder is

$$F=\frac{4\pi\mu U}{\ln\left(\dfrac{1}{\epsilon}\right)-1.497504970+2\epsilon-\epsilon^2/2-0.0025140724\epsilon^6+O(\epsilon^{12})}. \qquad [12]$$

For the high density approximation the walls of a channel are parts of three cylinders. In this case $c=3^{-1/2}l-a$. For a closed channel $l/2<a<3^{-1/2}l$,

$$w=\frac{P}{4\mu}\left[c^2-r^2+\frac{2(a+c)(c^3-r^3\cos 3\theta)}{3c(2a-c)}\right]$$

in the region

$$\left(-\frac{\pi}{3}<\theta<\frac{\pi}{3}\right),\quad 0<r<\frac{c}{\cos\theta}\left(1+\frac{c}{2a}\tan^2\theta\right),\quad Q=\frac{Pc^4}{\mu}\left(\frac{9\sqrt{3}}{20}+\frac{27\sqrt{3}}{40}\frac{c}{a}+O\left(\frac{c}{a}\right)^2\right)$$

and for two cells associated with each cylinder

$$F=\sqrt{3}Pl^2/2,\quad U=4Q/\sqrt{3}l^2,\quad \epsilon=2\pi a^2/\sqrt{3}l^2.$$

Hence

$$F=\frac{5\sqrt{3}\mu U}{2\left[1-\left(\dfrac{3\sqrt{3}\,\epsilon}{2\pi}\right)^{1/2}\right]^4\left[1+\dfrac{3}{2}\left(\left(\dfrac{2\pi}{3\sqrt{3}\epsilon}\right)^{1/2}-1\right)+\cdots\right]}. \qquad [13]$$

Numerical values from [12] and [13] are listed in Table 5 together with values from Happel's (1959) formula and Sparrow & Loeffler's (1959) figure 7. Equation [12] is very accurate for small ϵ and agrees within 1% with Sparrow & Loffler's values up to $\epsilon=0.7$.

Equation [13] agrees with these values at densities just below the touching density and should be better at higher densities. Happel's formula agrees well with the triangular grid results at low densities.

4. PARALLEL FLOW THROUGH A HEXAGONAL ARRAY

A typical sixth cell is $ABCD$ in (figure 3) and the total cell area is $3\sqrt{3}l^2/4$.

Half the cylinders have a neighbour in the positive x direction and half in the negative x direction so the flow pattern for each half is rotated through 180° relative to the pattern for the other half. There is a 120° periodicity in the fluid velocity which is

$$w = B''\ln\left(\frac{r}{a}\right) + \frac{P}{4\mu}(a^2 - r^2) + \sum_{n=1}^{\infty} B''_n\left[\left(\frac{a}{r}\right)^{3n} - \left(\frac{r}{a}\right)^{3n}\right]\cos 3n\theta.$$

For force balance

$$B'' = 3\sqrt{3}Pl^2/8\pi\mu.$$

We define

$$P''_{3n} = \sum_{p,q\neq 0,0}\sum \left(\frac{l}{d_{pq}}\right)^{3n} \quad \text{and} \quad \bar{P}''_{3n} = -1\sum_{p,q\neq 0,0}\sum \operatorname{sign}(p,q)\left(\frac{l}{d_{pq}}\right)^{3n}$$

where sign(p, q) is $+1$ or -1 if the grid point (p, q) has grid points around it arranged in the same way as the origin or turned through 180°. The d_{pq} are complex coordinates of all the grid points of the hexagonal net except the origin. The values of P''_{3n} and $\bar{P}''_{3n}$ are listed in Table 3.

On matching the two solutions we find that

$$B''_n = B''\left(\frac{a}{l}\right)^{3n}\left[\frac{P''_{3n}}{3n} - \frac{(3n+1)(3n+2)}{6}P''_3\bar{P}''_{3n+3}\left(\frac{a}{l}\right)^6 + \left[\frac{5(3n+1)(3n+2)}{3}P''_3\bar{P}''_6\bar{P}''_{3n+3} - \frac{(3n+1)\cdots(3n+5)}{6!}P''_6P''_{3n+6}\right]\left(\frac{a}{l}\right)^{12} + O\left(\left(\frac{a}{l}\right)^{18}\right)\right].$$

The flux over a cell around a cylinder is

$$\frac{27Pl^4}{32\pi\mu}\left[\ln\left(\frac{l}{a}\right) - 1.118384875 + \frac{4\pi a^2}{3\sqrt{3}l^2} - \frac{4\pi^2a^4}{27l^4} - \frac{P''^2_3}{3}\left(\frac{a}{l}\right)^6 + \left\{\frac{10}{3}P''^2_3\bar{P}''_6 - \frac{1}{6}P''^2_6\right\}\left(\frac{a}{l}\right)^{12} + O\left(\left(\frac{a}{l}\right)^{18}\right)\right].$$

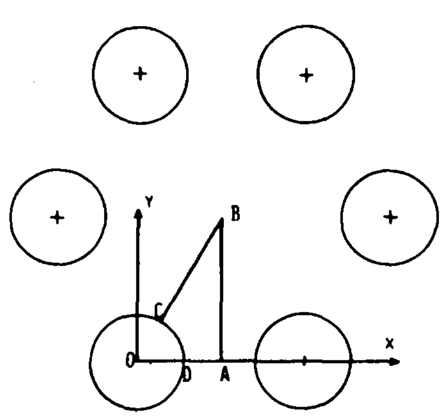

Figure 3. Hexagonal array.

Table 3. Values of P''_{3n} and $\bar{P}''_{3n}$.

3n	P''_{3n}	$\bar{P}''_{3n}$
3	2.75685 0788	
6	2.82294 1187	3.25723 9831
9	2.99324 1401	2.99324 1401
12	3.00894 1824	2.99245 4472
15	2.99990 9261	2.99990 9261
18	2.99970 6770	3.00031 6404
21	2.99999 8576	2.99999 8576
24	3.00001 1469	2.99998 8889
27	2.99999 9978	2.99999 9978
30	2.99999 9585	3.00000 0421
33	3.00000 0000	3.00000 0000
36	3.00000 0016	2.99999 9985
42	2.99999 9999	2.99999 9999

On replacing the last two calculated terms by a geometric series, and using $\epsilon = 4\pi a^2/3\sqrt{3}l^2$ the total force per unit length of a cylinder is

$$F = \frac{4\pi\mu U}{\ln\left(\frac{1}{\epsilon}\right) - 1.353663936 + 2\epsilon - \frac{\epsilon^2}{2} - 0.358221\epsilon^3/(1+2.26579\epsilon^3)}. \qquad [14]$$

For the high density approximation the walls of a channel are parts of six cylinders. In this case $c = l - a$. For a closed channel $l/2 < a < l$,

$$w = \frac{P}{4\mu}\left[c^2 - r^2 + \frac{(a+c)(c^6 - r^6\cos 6\theta)}{3c^4(5a-c)}\right]$$

in the region

$$\left(-\frac{\pi}{6} < \theta < \frac{\pi}{6}\right), \quad 0 < r < \frac{c}{\cos\theta}\left(1 + \frac{c}{2a}\tan^2\theta\right),$$

$$Q = \frac{Pc^4}{\mu}\left(\frac{293\sqrt{3}}{1134} + \frac{446\sqrt{3}}{10935}\frac{c}{a} + O\left(\left(\frac{c}{a}\right)^2\right)\right),$$

$$F = 3\sqrt{3}Pl^2/4, \qquad U = 2Q/(3\sqrt{3}l^2), \qquad \epsilon = 4\pi a^2/(3\sqrt{3}l^2).$$

Hence

$$F = \frac{7.54\mu U}{\left[1 - \left(\frac{3\sqrt{3}\epsilon}{4\pi}\right)^{1/2}\right]^4\left[1 + 0.158\left(\left(\frac{4\pi}{3\sqrt{3}\epsilon}\right)^{1/2} - 1\right) + \cdots\right]}. \qquad [15]$$

Numerical values from [14] and [15] are listed in Table 5 and agree moderately well near the touching density of $\pi/3\sqrt{3}$.

5. PARALLEL FLOW THROUGH A RECTANGULAR ARRAY

This calculation is done for one shape of rectangle with spacing l in the x direction and $2l$ in the y direction. A typical quarter cell is $ABCDE$ in (figure 4).

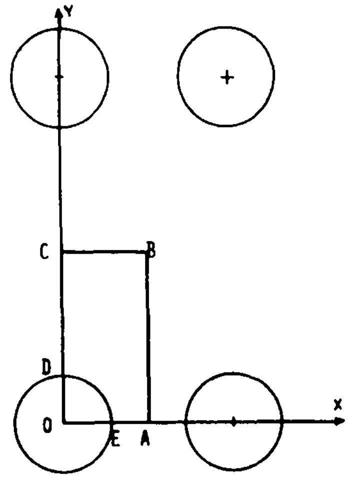

Figure 4. Rectangular array.

The general equation for the fluid velocity is

$$w = B''' \ln\left(\frac{r}{a}\right) + \frac{P}{4\mu}(a^2 - r^2) + \sum_{n=1}^{\infty} B_n'''\left[\left(\frac{a}{r}\right)^{2n} - \left(\frac{r}{a}\right)^{2n}\right]\cos 2n\theta.$$

For force balance $B''' = Pl^2/\pi\mu$.

We define

$$P_{2n}''' = \sum_{p,q \neq 0,0}\sum \left(\frac{l}{d_{pq}}\right)^{2n} = \sum_{p,q \neq 0,0}\sum (p + 2iq)^{-2n} \qquad [16]$$

and these are listed in Table 4.

On matching the two solutions we find that

$$B_n''' = B'''\left(\frac{a}{l}\right)^{2n}\left[\frac{P_{2n}'''}{2n} - \frac{2n+1}{2}P_2''' P_{2n+2}'''\left(\frac{a}{l}\right)^4 + O\left(\frac{a}{l}\right)^8\right].$$

The sum over the grid for P_2''' is shape dependent but the Green's integral over the outer

Table 4. Values of P_{2n}'''.

2n	P_{2n}'''	2n	P_{2n}'''
2	1.71879 6454	16	2.00006 5637
4	2.16645 8253	18	2.00000 0972
6	2.03110 9507	20	2.00000 3407
8	2.01151 7726	22	2.00000 0059
10	2.00014 2707	24	2.00000 0241
12	2.00115 8399	26	1.99999 9997
14	1.99995 0307	28	2.00000 0015
2n	$2 + \frac{2(1 + \cos n\pi)}{4^n} + \frac{4 \cos 2n \arctan 2}{5^n} + \ldots$		..

boundary is zero if the boundary is a central square. Also for flow symmetry at A

$$\sum_{n=1}^{\infty} \frac{P_{2n}'''}{4^n} = 1 - \frac{\pi}{8}.$$

This series was used to determine P_2'''.

The value of P_2''' may also be obtained by summing over the grid using [16] extending equally in the x and y directions, or taking twice as many p's as q's. The sum is positive because the density of grid points in the y direction is half the density in the x direction but the convergence is rather slow.

The flux over the rectangle associated with the cylinder is

$$\frac{2Pl^4}{\pi\mu}\left\{\ln\left(\frac{l}{a}\right) + \frac{1}{2}\ln\left(\frac{5}{4}\right) - \frac{3}{2} - \frac{3}{4}\arctan 2 + \frac{19\pi}{48} - \sum_{n=1}^{\infty} \frac{(5/4)^{n+1} P_{2n}''' \sin(n+1)\arccos(-0.6)}{2n(n+1)(2n+1)} + \frac{\pi a^2}{l^2} - \frac{\pi^2 a^4}{16 l^4} + P_2'''\left(\frac{a}{l}\right)^4\left[\frac{\pi}{4} - \arctan 2 + \sum_{n=2}^{\infty} \frac{(5/4)^n P_{2n}''' \sin n \arccos(-0.6)}{2n}\right] + O\left(\frac{a}{l}\right)^6\right\}.$$

The two series involved in this calculation are divergent but are summed by subtracting comparison series

$$-\mathscr{I} \sum_{n=1}^{\infty} \frac{(-0.75+i)^{n+1}}{n(n+1)(2n+1)} \quad \text{and} \quad \mathscr{I} \sum_{n=2}^{\infty} \frac{(-0.75+i)^n}{n}.$$

where $\mathscr{I}$ means the "imaginary part of".

The flux is

$$\frac{2Pl^4}{\pi\mu}\left\{\ln\left(\frac{l}{a}\right) - 0.79067254 + \frac{\pi a^2}{2l^2} - \frac{\pi^2 a^4}{16 l^4} - \frac{P_2'''^2}{2}\left(\frac{a}{l}\right)^4 + O\left(\left(\frac{a}{l}\right)^6\right)\right\}.$$

Table 5. Values of $F/\mu U$ for parallel flow for various grids and formulae. S & L means Sparrow & Loeffler, (figure 7).

	F/μU								
	Happel's formula	Triangular grid			Square grid			Hex. grid	
ε		Eq12	Eq13	S & L	Eq10	Eq11	S & L	Eq14	Eq15
0	0	0		4.4	0		4.4	0	
0.1	12.6	12.6		12.4	12.3		12.4	11.0	
0.2	25.7	25.5		26.3	24.5		24.5	19.8	
0.3	48.5	48.1		48.1	44.5		43.9	31.7	
0.4	92.2	90.6		90.7	79.2		78.5	47.9	49.5
0.5	184	178		177.2	141	149	142	70.0	71.7
0.6	408	378	348	383	257	268	258	100.4	102.5
$\pi/3\sqrt{3}$								102.0	104.2
0.7	1076	906	900	902	476	488	479	141	146
$\pi/4$					821	836	830		
0.8	3998	2524	2656	2660	901	920			207
0.9	34900	8270	9844	10300		1830			296
$\pi/2\sqrt{3}$		9069	10909	11300					
1.0	∞		55900			3036			427
$2\pi/3\sqrt{3}$			∞						
$\pi/2$						∞			
$4\pi/3\sqrt{3}$									∞

and the total force per unit length of the cylinder is

$$F = \frac{4\pi\mu U}{\ln(1/\epsilon) - 1.12976238 + 2\epsilon - 1.197317\epsilon^2 + O(\epsilon^3)}.$$

This is the most irregular arrangement allowing enhanced flow through the wider channels.

6. CONCLUSION FOR PARALLEL FLOW

The drag calculations for the four arrangements of cylinders have an order of accuracy ranging from ϵ^2 for the rectangular array to ϵ^6 for the triangular array and may be used from low to moderately high densities. They also match up reasonably well with the high density calculations and with Sparrow & Loeffler's results.

It must be emphasised that these are regular arrangements which completely ignore boundary or wall effects and asymmetry of any vessel containing the cylinders.

At very low densities, when $\ln(1/\epsilon)$ is large, Happel's simple cell model is a good approximation for the triangular grid.

With zero drag on the outer cell wall the force balance gives the constant, B, in the general solution such as [1]. Thus $B = P \times$ (cell area)$/2\pi\mu$ independent of the arrangement of cylinders. Furthermore the velocity near a cylinder is

$$w \doteqdot \frac{P \times (\text{cell area})}{2\pi\mu} \ln\left(\frac{r}{a}\right) + \frac{P}{4\mu}(a^2 - r^2).$$

The drag on unit length of a cylinder when the superficial speed is U and the density of the cylinders is ϵ is $F \doteqdot 2\pi\mu U/(\ln(1/\epsilon) - K)$ where K is close to 1.5. This suggests that the cylinders may be regarded as drag elements almost independent of their geometric arrangement and dependent only on their volume concentration. In the more refined calculation K is a maximum for the most compact triangular arrangement which is very close to the value given by Happel's (1959) model while a smaller value of K occurs when the gaps have a range of sizes. Thus a reasonably accurate empirical formula for the drag force may be constructed in the form

$$F = \frac{4\pi\mu U}{\ln(1/\epsilon) - K + 2\epsilon - \frac{1}{2}\epsilon^2}$$

where K is determined experimentally and is a measure of the regularity of the arrangement.

7. TRANSVERSE FLOW PARALLEL TO THE SIDES OF A SQUARE GRID

Description

We use the same arrangement of cylinders as in section 2 and appendix 1.

Let the fluid have a mean velocity U in the x direction with zero velocity on the cylinder walls and be driven by a pressure gradient $-P$, in the x direction. Let the fluid flow satisfy the Stokes equations,

$$\operatorname{div} \mathbf{v} = 0, \quad \operatorname{grad} p = \mu \nabla^2 \mathbf{v}.$$

If $\mathbf{v} = \operatorname{curl}(0, 0, \chi)$ then $\nabla^2 p = 0$ and $\nabla^4 \chi = 0$.

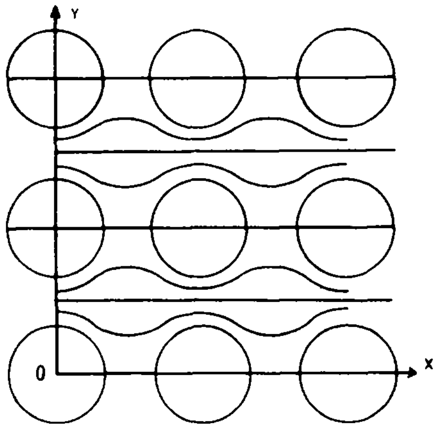

Figure 5. General transverse flow for a square array.

First solution

The general solution in polar (r, θ) coordinates symmetric about the y-axis, anti-symmetric about the x-axis, bounded in $a \le r < l$ with zero velocity when $r = a$ is

$$p = p_0 + \sum_{n=0}^{\infty}\left[A_n\left(\frac{r}{a}\right)^{2n+1} + C_n\left(\frac{a}{r}\right)^{2n+1}\right]\cos(2n+1)\theta, \qquad [17]$$

and

$$\begin{aligned}\chi = {} & \frac{a^2}{8\mu}\left\{A_0\left[\left(\frac{r}{a}\right)^3 - 2\frac{r}{a} + \frac{a}{r}\right] + C_0\left[\frac{-4r}{a}\ln\left(\frac{r}{a}\right) + \frac{2r}{a} - \frac{2a}{r}\right]\right\}\sin\theta \\ & + \frac{a^2}{8\mu}\sum_{n=1}^{\infty}\left\{A_n\left[\frac{-1}{n+1}\left(\frac{r}{a}\right)^{2n+3} - \frac{2}{2n+1}\left(\frac{r}{a}\right)^{2n+1} + \frac{1}{(n+1)(2n+1)}\left(\frac{a}{r}\right)^{2n+1}\right]\right. \\ & \left. + C_n\left[\frac{-1}{n(2n+1)}\left(\frac{r}{a}\right)^{2n+1} + \frac{1}{n}\left(\frac{a}{r}\right)^{2n-1} - \frac{2}{2n+1}\left(\frac{a}{r}\right)^{2n+1}\right]\right\}\sin(2n+1)\theta, \qquad [18]\end{aligned}$$

with polar velocity components

$$v_r = \frac{1}{r}\frac{\partial\chi}{\partial\theta}, \quad v_\theta = -\frac{\partial\chi}{\partial r}.$$

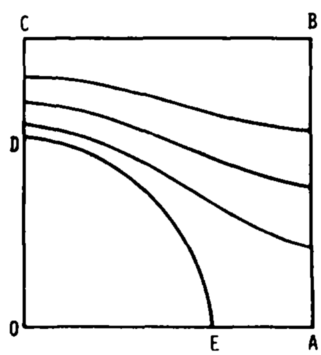

Figure 6. Typical quarter cell for a square array.

Second solution

This is justified in appendix 2. If the boundary conditions are such as to produce a periodic flow pattern then the pressure and stream function will have identical singularities inside every cylinder with constant and linear terms determined by the outer mechanism producing the flow.

Let P be a point in the flow field with polar coordinates (r, θ) and complex coordinate $z = re^{i\theta}$. Let the centre of each cylinder with coordinates (pl, ql) have a complex coordinate $d_{pq} = b_{pq}\,e^{i\alpha_{pq}} = (p + iq)l$ where p and q are integers. Let the complex coordinate of the point P referred to this point be $z_{pq} = r_{pq}\,e^{i\theta_{pq}}$, then $z_{pq} = z - d_{pq}$. The second solution can be written as

$$p = p_0 + \mathscr{R}\left\{Kz + \sum_{n=0}^{\infty}\sum_{p}\sum_{q} C_n(a/z_{pq})^{2n+1}\right\}. \qquad [19]$$

$$\chi = \mathscr{I}(\chi_0 + Hz) + \mathscr{I}\frac{a^2}{8\mu}\sum_{p}\sum_{q}\left\{-A_0\frac{a}{z_{pq}} + C_0\left(-\frac{2z_{pq}}{a}\ln\left(\frac{z_{pq}\bar{z}_{pq}}{b_{pq}^2}\right) + \frac{2a}{z_{pq}}\right)\right.$$
$$\left. + \sum_{n=1}^{\infty}\left[\frac{-A_n}{(n+1)(2n+1)}\left(\frac{a}{z_{pq}}\right)^{2n+1} + C_n\left[\frac{-\bar{Z}_{pq}}{an}\left(\frac{a}{z_{pq}}\right)^{2n} + \frac{2}{2n+1}\left(\frac{a}{z_{pq}}\right)^{2n+1}\right]\right]\right\}. \qquad [20]$$

where $\mathscr{R}$ stands for "real part of", $\mathscr{I}$ for "imaginary part of" and K, χ_0 and H depend on the outer integral.

We now equate [17] and [19] for the pressure and cancel those singular terms which come from the central cylinder leaving

$$\sum_{n=0}^{\infty} A_n(z/a)^{2n+1} = Kz + \sum_{m=0}^{\infty}\sum_{p,q \neq 0,0}\sum C_m\left(\frac{a}{z - d_{pq}}\right)^{2m+1} \qquad [21]$$

This is now true for all z and not just the real part.

The r.h.s. may be written as a power series in z convergent for $|z| < |\text{least } d_{pq}| = l$ and equated term by term to the left hand series.

As in the case of parallel flow we define

$$P_n = \sum_{p,q \neq 0,0}\sum\left(\frac{l}{d_{pq}}\right)^n = \sum_{p,q \neq 0,0}\sum\left(\frac{l}{b_{pq}}\right)^n e^{-ni\alpha_{pq}} = \sum_{p,q \neq 0,0}\sum (p + iq)^{-n}$$

and in addition

$$Q_n = \sum_{p,q \neq 0,0}\sum\frac{\bar{d}_{pq}l^{n-2}}{d_{pq}^{n-1}} = \sum_{p,q \neq 0,0}\sum\left(\frac{l}{b_{pq}}\right)^{n-2} e^{-ni\alpha_{pq}} = \sum_{p,q \neq 0,0}\sum (p - iq)(p + iq)^{-n+1}. \qquad [22]$$

For a symmetric grid $P_{2n+1} = Q_{2n+1} = 0$ while P_{2n} and Q_{2n} are real. Hence the coefficients of $(z/a)^{2n+1}$ in [21] give the set of relations

$$A_n = -\sum_{m=0}^{\infty} C_m\frac{(2m + 2n + 1)!}{(2m)!(2n + 1)!}P_{2m+2n+2}\left(\frac{a}{l}\right)^{2m+2n+2} \qquad n = 0, 1, 2, \ldots, \qquad [23]$$

if we absorb K into A_0 and adjust P_2 accordingly.

Similarly, if we write [18] as the imaginary part of a complex function of $\bar{z}$ and powers of z, expand out [20] and equate coefficients of $\bar{z}z^{2n+1}$ we recover [23] again. The coefficient of z^{2n}, however, gives us

$$C_n = -C_0 P_{2n}\left(\frac{a}{l}\right)^{2n} - \sum_{m=0}^{\infty} C_m \frac{(2m+2n)!}{(2n-1)!(2m)!}\left(\frac{a}{l}\right)^{2m+2n} Q_{2m+2n+2}$$
$$+ \sum_{m=0}^{\infty}\left[C_m \frac{2n}{(2n+1)!}\frac{(2m+2n+2)!}{(2m+1)!} - A_m \frac{(2m+2n+1)!}{(2n-1)!(2m+2)!}\right]$$
$$\times \left(\frac{a}{l}\right)^{2m+2n+2} P_{2m+2n+2} + \cdots$$
$$n = 1, 2, 3, \ldots \qquad [24]$$

Equations [23] and [24] may be solved step by step in successive powers of (a/l) to express A_n and C_n as multiples of C_0 giving us

$$\frac{A_{n-1}}{C_0}\left(\frac{l}{a}\right)^{2n} = -P_{2n} + n(2n+1)(P_2 + 2Q_4)P_{2n+2}\left(\frac{a}{l}\right)^4 - 8n(2n+1)P_4 P_{2n+2}\left(\frac{a}{l}\right)^6$$
$$+\left[\frac{n(n+1)(2n+1)(2n+3)}{6}(P_4 + 4Q_6)P_{2n+4}\right.$$
$$\left. - 3n(2n+1)P_{2n+2}(P_2P_4 + 4(P_2 + 2Q_4)Q_6\right]\left(\frac{a}{l}\right)^8 + O\left(\frac{a}{l}\right)^{10} \qquad [25]$$

and

$$\frac{C_n}{C_0}\left(\frac{l}{a}\right)^{2n} = -(P_{2n} + 2nQ_{2n+2}) + 4n(n+1)P_{2n+2}\left(\frac{a}{l}\right)^2 + [n(2n+1)P_2P_{2n+2}$$
$$+ 2n(n+1)(2n+1)(P_2 + 2Q_4)Q_{2n+4}]\left(\frac{a}{l}\right)^4 - \left[16n(n+1)(2n+1)P_4Q_{2n+4}\right.$$
$$\left. + \frac{4n(n+1)(n+2)(2n+3)}{3}(P_2 + 2Q_4)P_{2n+4}\right]\left(\frac{a}{l}\right)^6$$
$$+\left[\frac{n(n+1)(2n+3)(22n+43)}{2}P_4P_{2n+4} + \frac{n(n+1)(n+2)(2n+1)(2n+3)}{3}\right.$$
$$\times (P_4 + 4Q_6)Q_{2n+6} - 3n(2n+1)(P_2 + 2Q_4)P_4P_{2n+2}$$
$$\left. - 6n(n+1)(2n+1)(P_2P_4 + 4(P_2 + 2Q_4)Q_6\, Q_{2n+4}\right]\left(\frac{a}{l}\right)^8 + O\left(\frac{a}{l}\right)^{10} \qquad [26]$$

8. DETERMINATION OF P_n AND Q_n

P_n and Q_n are defined by [5] and [22]. For a square grid P_2 and Q_4 are indeterminate because they depend on the order of summation when we neglect the outer integral. They may be deduced indirectly from symmetries of the pressure and velocity fields or by extending the grid infinitely more perpendicular than parallel to the flow. The other P_n and Q_n are all zero if n is not a multiple of 4. The non-zero values are listed in Table 1.

Apart from P_2, the P_n's are the same as for parallel flow. The early P_n's and Q_n's may be evaluated by summing first over p or q. P_4, P_8 and P_{12} have been given previously and, if $S_q = \sinh^2 q\pi$, then

$$P_2 = -\frac{\pi^2}{3} + 2\sum_{q=1}^{\infty}\frac{\pi^2}{S_q} = -\pi,$$

$$Q_4 = \frac{\pi^2}{3} + 2\pi^2 \sum_{q=1}^{\infty} \frac{2q\pi \coth q\pi - 1}{S_q},$$

$$Q_8 = \frac{2\pi^6}{945} + \frac{2\pi^6}{45} \sum_{q=1}^{\infty} \frac{2q\pi \coth q\pi (2S_q^2 + 30S_q + 45) - 6S_q^2 - 45S_q - 45}{S_q^3}.$$

The other Q_n's are obtained by direct summation over the grid.

9. CALCULATION OF C_0

Consider the x-component of the Stokes equation,

$$\partial p/\partial x = \mu \nabla^2 u$$

and

$$\iint_S (\partial p/\partial x)\,\mathrm{d}x\,\mathrm{d}y = \iint_S \mu \nabla^2 u\,\mathrm{d}x\,\mathrm{d}y$$

where S is the interior of the section $ABCDE$ in (figure 6). On integrating over one coordinate, this converts to a boundary integral, balancing pressure and shear forces.

By using [17] and [18] and the same method as for finding B in [8] we obtain

$$C_0 = -Pl^2/2\pi a. \qquad [27]$$

10. CALCULATION OF P_2 AND Q_4

P_2 and Q_4 are defined as

$$P_2 = \sum\sum \frac{l^2 \cos 2\alpha_{pq}}{b_{pq}^2} = \sum_{p,q \neq 0,0}\sum \frac{p^2 - q^2}{(p^2 + q^2)^2}$$

and

$$Q_4 = \sum\sum \frac{l^2 \cos 4\alpha_{pq}}{b_{pq}^2} = \sum_{p,q \neq 0,0}\sum \frac{p^4 - 6p^2q^2 + q^4}{(p^2 + q^2)^3}.$$

The first sum is indeterminate, ranging from $-\pi$ if we sum over q first to $+\pi$ if we sum over p first. The second is also indeterminate ranging from 4.08 . . if the sum is taken over p or q first to -0.94 . . if the grid is first summed at 45° to the axis.

Lord Rayleigh (1892) first encountered this difficulty in his conductivity calculations. He chose $P_2 = -\pi$ and used a physical argument to justify his choice. However, we can find P_2 and Q_4 by using two symmetries of the flow field.

By symmetry, the pressure at $A(r = l/2, \theta = 0)$ is $p_0 - Pl/2$. Hence, on substituting the coordinates of A in [17] for the pressure and using [25] and [26] we obtain

$$-\frac{Pl}{2} = \frac{2C_0 a}{l}\left[1 - \frac{P_2}{4} - \sum_{n=1}^{\infty} \frac{P_{4n}}{16^n}\right].$$

On substituting for C_0 from [27], we get

$$P_2 = -2\pi + 4 - 4\sum_{n=1}^{\infty} (P_{4n}/16^n) = -\pi$$

to at least 10 decimal places.

From considerations dealt with in appendix 2 the boundary integral contributes nothing to P_2 if the grid forms an infinite slab perpendicular to the flow. We may, therefore, obtain the same value for P_2 by using the original definition and summing over q first before p. This results in the expression for P_2 given near the end of section 8.

We may determine Q_4 from one of the velocity symmetries at A, B or C where the velocity components have either maxima or saddle points. To obtain a rapidly convergent series for Q_4 we use the saddle point of v_r at $A(r = l/2, \theta = 0)$, which gives

$$r\frac{\partial v_r}{\partial r} = r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \chi}{\partial \theta}\right) = 0.$$

On substituting [18], [25] and [26] in this equation we find that it is satisfied if

$$Q_4 = 4 - \sum_{n=1}^{\infty}(2n+1)Q_{4n+4}16^{-n} + \sum_{n=1}^{\infty}(2n-1)P_{4n}4^{1-2n}$$
$$= 4.0784511611614\,..$$

This number can also be found from the original definition of Q_4 if we sum over the grid, first completely over p then q, or q then p, but not in any other order. The resulting expression for Q_4 is given at the end of section 8.

11. THE FLUX AND AVERAGE VELOCITY

The flux through the representative quarter cell in figure 6 is the flux across DC or AB. The stream function, χ, is zero along DEA so the flux is the value of χ from [18] at $C(r = l/2, \theta = \pi/2)$ or at $B(r = l/\sqrt{2}, \theta = \pi/4)$. If we substitute for A_n and C_n from [25] and [26] the resulting series at C are more rapidly convergent than the corresponding series at B. The mean fluid velocity across the whole cell is obtained by dividing the flux by the quarter cell width $l/2$. The mean fluid velocity so calculated is

$$\frac{Pl^2}{4\pi\mu}[\ln(l/a) - 1.310532927 + \pi a^2/l^2 - 8.75573387(a/l)^4 + 63.21721610(a/l)^6$$
$$- 235.8407557(a/l)^8 + O(a/l)^{10}].$$

This is found to agree very closely with

$$\frac{Pl^2}{4\pi\mu}\left[\ln(l/a) - 1.310532927 + \pi\left(\frac{a}{l}\right)^2 - \frac{(P_2+2Q_4)^2 + P_2^2}{4}\left(\frac{a}{l}\right)^4 + 4P_4(P_2+2Q_4)\left(\frac{a}{l}\right)^6 - \frac{19}{4}P_2P_4(P_2+2Q_4)\left(\frac{a}{l}\right)^8 + O\left(\frac{a}{l}\right)^{10}\right]. \quad [28]$$

As the density of cylinders is $\pi a^2/l^2$ the mean velocity is

$$U = (F/8\pi\mu)[\ln(1/\epsilon) - 1.47633597 + 2\epsilon - 1.77428264\epsilon^2 + 4.07770444\epsilon^3 - 4.84227402\epsilon^4 + O(\epsilon^5)] \quad [29]$$

where $F = Pl^2$ is the force on unit length of a cylinder.

The first three terms in [29] are half the corresponding terms for the parallel flow. They agree with Hasimoto's (1959) and Sangani & Acrivos's (1982b) terms and show that Kuwabara's & Happel's approximations are also good at low densities.

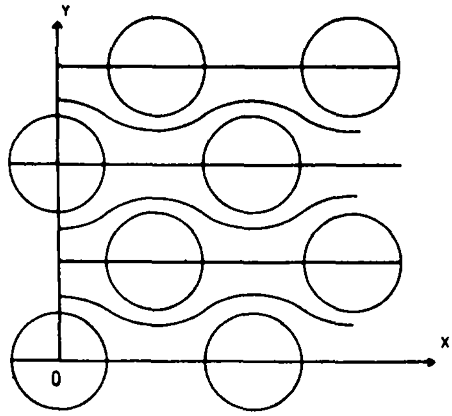

Figure 7. General transverse diagonal flow for a square array.

Equation [29] adds one more term to Sangani & Acrivos's low density formula and gives the coefficients to eight decimal instead of three decimal accuracy. It may also be improved by replacing the last four or five terms by their Padé approximant. Hence

$$U = \frac{F}{8\pi\mu}\left[\ln(1/\epsilon) - 1.47633597 + \frac{2\epsilon - 0.79589781\epsilon^2}{1 + 0.48919241\epsilon - 1.60486942\epsilon^2}\right]. \qquad [30]$$

This is very accurate for small ϵ, agrees with Sangani & Acrivos's (1982a) collocation calculations within 0.3% when $\epsilon = 0.2$, 4% when $\epsilon = 0.3$ and deteriorates rapidly thereafter.

12. DIAGONAL TRANSVERSE FLOW THROUGH A SQUARE GRID

The tensor form of Darcy's equation is $u_i = a_{ij}\partial p/\partial x_j$. Using symmetry arguments it can be shown that for a square grid $a_{12} = 0$ and $a_{11} = a_{22}$. Hence the Darcy law permeability is isotropic for a square grid.

For flow at 45° to a square grid a typical half cell of the flow is $OABC$ in (figure 8) where $OB = l$. With the exception of [22], [17]–[26] apply. For P_n and Q_n in [22], the grid of cylinder centres has been rotated through 45° so P_{in} and Q_{8n} in Table 1 are unaltered and P_{8n+4}, Q_{8n+4} reverse sign. Integration of the Stokes equation over the half cell $EAGFCD$ in (figure 8) for the force balance again gives us $C_0 = -Pl^2/2\pi a$. By symmetry the pressure at H is $-Pl/2\sqrt{2}$. If we calculate this, as before, from [17], [25]–[27] we find that P_2 is again $-\pi$. Similarly, if $\partial v_r/\partial r = 0$ at H and A while $\partial v_\theta/\partial r = 0$ at H and C, we find that $(P_2 + 2Q_4)$ reverses sign. Hence for diagonal flow

$$Q_4 = -0.936858510.$$

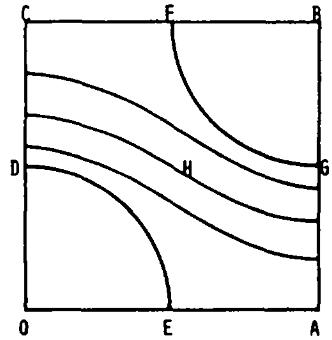

Figure 8. Typical half cell for a diagonal square array.

The mean fluid velocity across the half cell in (figure 8) is $\sqrt{2}\chi_C/l$ or $2\sqrt{2}\chi_H/l$. The series for the second expression is more rapidly convergent and the calculated mean velocity is identical, at least to order $(a/l)^8$, with the mean velocity parallel to the sides of the square array as calculated in section 11.

13. FLOW ACROSS A TRIANGULAR ARRAY

We first consider an array whose cross section is an equilateral triangular grid with flow in the direction of a nearest neighbour, as in (figure 9). A typical quarter cell of the flow is $OABC$ in (figure 10). In calculating P'_n and Q'_n, the centres of the cylinders with spacing l have coordinates $[(p+q/2)l, \sqrt{3}ql/2]$. The grid is unchanged by a rotation of 60°. Hence, apart from P'_2 and Q'_4, P'_n and Q'_n are zero unless n is a multiple of 6. P'_2 and O'_4 can be deduced by using the symmetries of pressure and velocities at A, H or C. A and H being equidistant and closest to 0 yield more rapidly convergent series.

The force balance integral gives

$$C_0 = -\sqrt{3}Pl^2/4\pi a.$$

Calculation of the pressures at A and H gives a series,

$$P'_2 = -4\pi/\sqrt{3} + 4 - 4\sum_{n=1}^{\infty} P'_{6n}/64^n = -2\pi\sqrt{3}.$$

Futhermore, $\partial v_r/\partial r = 0$ at A and H if $Q'_4 = \pi/\sqrt{3}$.

The values of P'_{6n} and Q'_{6n} are found by summing a double series similar to [22] over the triangular grid. P'_{6n} are the same as for longitudinal flow. The values of P'_2 and Q'_4 may also be obtained by summing over q before $(p+q/2)$. The values of P'_n and Q'_n are listed in Table 2.

Let $C_q = \cosh^2 \pi\sqrt{3}q/2$ and $S_q = \sinh^2 \pi\sqrt{3}q/2$ then

$$P'_2 = -\frac{\pi^2}{3} - 2\pi^2 \sum_{q\,\text{odd}} \frac{1}{C_q} + 2\pi^2 \sum_{q\,\text{even}} \frac{1}{S_q} = -2\pi/\sqrt{3},$$

$$P'_6 = \frac{2\pi^6}{945} + \frac{2\pi^6}{15}\sum_{q\,\text{odd}} \frac{15 - 15C_q + 2C_q^2}{C_q^3} - \frac{2\pi^6}{15}\sum_{q\,\text{even}} \frac{15 + 15S_q + 2S_q^2}{S_q^3},$$

$$Q'_4 = \frac{\pi^2}{3} - 2\pi^2 \sum_{q\,\text{odd}} \frac{\pi\sqrt{3}q\tanh(\pi\sqrt{3}q/2) - 1}{C_q} + 2\pi^2 \sum_{q\,\text{even}} \frac{\pi\sqrt{3}q\coth(\pi\sqrt{3}q/2) - 1}{S_q},$$

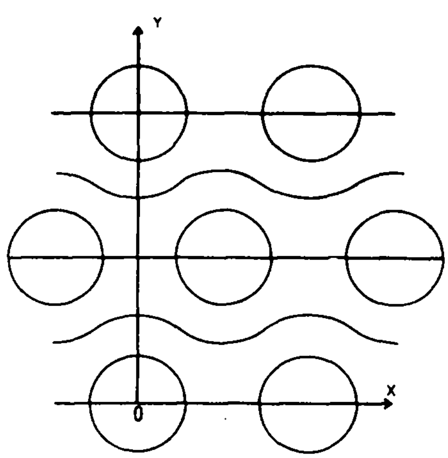

Figure 9. General transverse flow for a triangular array.

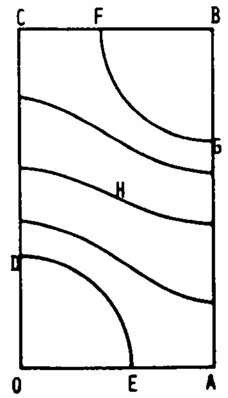

Figure 10. Typical half cell for a triangular array.

$$Q_6' = \frac{\pi^4}{45} + \frac{2\pi^4}{3} \sum_{q\,\mathrm{odd}} \frac{\pi\sqrt{3}q \tanh(\pi\sqrt{3}q/2)(C_q - 3) - 2C_q + 3}{C_q^2}$$
$$+ \frac{2\pi^4}{3} \sum_{q\,\mathrm{even}} \frac{3 + 2S_q - \pi\sqrt{3}q \coth(\pi\sqrt{3}q/2)(3 + S_q)}{S_q^2}.$$

The mean velocity across the cell is $2\chi_C/\sqrt{3}l$ or $4\chi_H/\sqrt{3}l$ and is given by

$$U = \frac{F}{4\pi\mu}\{\ln(l/a) - 1.393037947 + 3.62759878\,(a/l)^2$$
$$- 3.289868133\,(a/l)^4 - 63.99883744\,(a/l)^8$$
$$+ 795.9843482\,(a/l)^{10} - 3683.869236\,(a/l)^{12} + O(a/l)^{14}\}. \quad [31]$$

The last five coefficients are very close to $2\pi/\sqrt{3}$, $\pi^2/3$, $2Q_6'^2$, $24P_6'Q_6'$ and $(10P_2'Q_6' - 433P_6'/6)P_6'$ respectively.

As the density of the cylinders is $2\pi a^2/\sqrt{3}l^2$ and the last three terms appear to belong to an almost geometric series we replace the last two by a geometric series. Then

$$U = \frac{F}{8\pi\mu}\left\{\ln\left(\frac{1}{\epsilon}\right) - 1.497504972 + 2\epsilon - \frac{\epsilon^2}{2} - 0.739137296\epsilon^4 + \frac{2.534185018\epsilon^5}{1 + 1.275793652\epsilon} + \cdots\right\}. \quad [32]$$

The first four terms are half the corresponding terms for longitudinal flow while the second term disagrees slightly with that of Sangani & Acrivos (1982b) and three new terms have been added to their result.

This formula is very accurate for small ϵ, agrees with Sangani & Acrivos's (1982a) collocation calculations within 0.1% when $\epsilon = 0.3$, 1.3% when $\epsilon = 0.4$, 10% when $\epsilon = 0.5$ and deteriorates rapidly thereafter.

14. FLOW DIRECTION TURNED THROUGH 30°

This flow direction bisects lines joining two nearest neighbours and is illustrated in (figures 11 and 12). The rotation of the grid changes the signs of P'_{12n+6} and Q'_{12n+6} and leaves P_2', Q_4', P'_{12n} and Q'_{12n} unaltered. These changes alter the pressure and velocity fields but the mean velocity, as given by [31], is unchanged.

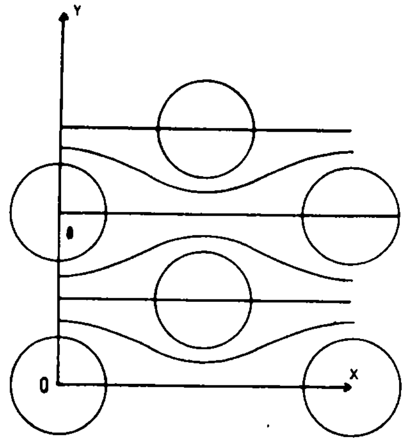

Figure 11. Second flow pattern for a triangular array.

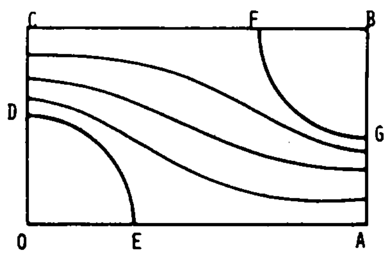

Figure 12. Second typical half cell for a triangular array.

15. CONCLUSION

The most compact arrangement of cylinders is the equilateral triangular array. By comparing [29] and [32] for small densities we see that the mean velocity through a triangular grid is slightly less than the mean velocity through a square grid of equal density. On the other hand when the cylinders touch flow ceases for the square array when $l = 2a$ or the density of cylinders is $\pi/4$ and it does not cease for the triangular array till a higher value of the density is reached.

Of the two simple cell models that of Kuwabara is better.

If we compare the transverse and longitudinal flow we see that the drag for transverse flow is twice the drag for longitudinal flow with a deviation of order $(a/l)^4$ for the square array and $(a/l)^8$ for the triangular array.

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APPENDIX 1

The method of singularities for parallel flow through a square array.

Following O'Brien (1979) we may set up Green's integrals for the pressure and velocity fields and by using two different domains set up two equivalent sums for the pressure and velocity. These two sums can be equated to deduce the unknown coefficients in the sums.

Description

Let the medium contain solid cylinders of radius a with axes parallel to the z-axis arranged in a square lattice with centre lines cutting the xy-plane at points with coordinates (pl, ql) where p and q are integers. For separate cylinders a must be less than $l/2$.

In a real flow situation the cylinders would form a finite cloud or be contained within the walls of a tank. However to avoid these wall or boundary effects we assume the solution is completely periodic. This can be achieved by containing the cylinders or their circular cross-sections within four slippery walls at $x=(I+\frac{1}{2})l$, $x=-(J+\frac{1}{2})l$, $y=(K+\frac{1}{2})l$, $y=-(L+\frac{1}{2})l$ where I, J, K and L are integers. If w is the fluid velocity in the z direction and n is the outward normal, $\partial w/\partial n = 0$ on these walls.

If this solution is to be tested inside a tank with no-slip walls, such walls would need to be positioned carefully somewhere near the next outward row of cylinders so as to cause minimum disturbance to the flow around all but the outer layer of cylinders.

The Stokes flow equations reduce to the two-dimensional equation

$$\nabla^2 w = -P/\mu$$

in the region between the circles and the outer rectangle.

Using rectangular and polar coordinates, let P be a fixed point with coordinates (x_0, y_0) or (r_0, θ_0) in the flow region just outside the circle centred at the origin. Let Q be a variable point with coordinates (x, y) and (r, θ) or with coordinates (r_i, θ_i) with respect to each of the grid points with address i and polar coordinates (b_i, β_i). The boundary conditions are $w=0$ on each circle and $\partial w/\partial n = 0$ on the rectangle. Hence uniqueness is guaranteed by the uniqueness theorem for the Laplace equation.

From periodicity and square symmetry, w is of the form $\Sigma_{n=0}^{\infty} w_n \cos 2n\pi x/l$ when y is $(K+\frac{1}{2})l$ or $-(L+\frac{1}{2})l$ and $\Sigma_{n=0}^{\infty} w_n \cos 2n\pi y/l$ when x is $(I+\frac{1}{2})l$ or $-(J+\frac{1}{2})l$ and $\partial w/\partial n$ is of the form $\Sigma_{n=0}^{\infty} a_n \cos 4n\theta_i$ and is the same on every circle. Furthermore the constants a_n and w_n are unique.

We may now find two expressions for w using a simple Green's function,

$$G(x, y, x_0, y_0) = \frac{1}{2\pi} \ln \rho \quad \text{where} \quad \rho^2 = (x-x_0)^2 + (y-y_0)^2$$

and the Lagrange identity

$$w\nabla^2 G - G\nabla^2 w = \operatorname{div}(w \operatorname{grad} G - G \operatorname{grad} w).$$

When integrating over the flow region V with an outer surface S_R and cylinder surfaces S_i with

$$\nabla^2 G = \delta(x-x_0)\delta(y-y_0) \quad \text{and} \quad \nabla^2 w = -P/\mu,$$

we divide V into its basic square cells around each of the circles and call these V_i. Hence

$$w(x_0, y_0) = \frac{1}{2\pi} \int_{S_R} w \frac{\partial \ln \rho}{\partial n} \, dS_R - \sum_i \left\{ \iint_{V_i} \frac{P \ln \rho}{2\pi\mu} \, dV_i + \int_{S_i} \frac{\ln \rho}{2\pi} \frac{\partial w}{\partial n_i} \, dS_i \right\}.$$

In the integrals over each circle and its associated square, the dominant term from the circle is $-a_0 a \ln b_i$ while $-P(l^2-a^2)\ln b_i/2\pi\mu$ dominates the surface integral. The sum of these terms will not diverge if $2\pi a\mu a_0 = -P(l^2 - \pi a^2)$.

The physical interpretation of this condition is that the shear on each cylinder is in equilibrium with the fluid thrust on its associated cell.

The integral over the outer rectangle is equal to w_0 plus smaller terms.

If x_0, y_0, $(I-J)$ and $(K-L)$ are non-zero the integrals will contain linear and quadratic terms in these variables. However, since $w(x_0, y_0)$ is unique its value will be independent of the shape of the region over which we integrate. Hence the integral over the outer rectangle will cancel any shape dependent terms from the other integrals and may be interpreted as the sum of contributions from additional circles and cells and the sources of pressure and flow outside the rectangle.

If we wish to eliminate the integral over the outer rectangle we make it into a large square with $I = J = K = L$ and let I tend to infinity leaving us with an unknown constant, a term $-P(x_0^2, y_0^2)/4\mu$ and a sum of harmonic terms with singularities of the same type and size at the centre of each circle. Their sum will converge if the pressure and shear forces balance.

We get a second solution if we contract the rectangle to a single cell of side l with $I = J = K = L = 0$. For the Green's integral around the inner circle and the surface up to radius r_0, if $r_0 < l/2$, ρ may be expanded as a power series in (r/r_0) giving the same singular terms at the origin of r_0 as before. For the rest of the surface integral and boundary integral for $r > r_0$, ρ is expanded as a power series in (r_0/r). Hence the contributions from sources outside r_0 reduce to a set of harmonics with positive powers of r_0. These combine together to form a Laurent series.

The restriction $r_0 > l/2$ is artificial because the Green's integral can be carried out with any shape of contour. If we stretch the outer square with appropriate changes to w and $\partial w/\partial n$ on the boundary, the first singularities are encountered at the centres of the four neighbouring circles at radius l. Hence the Laurent series will converge between radius a and radius l.

Finally, $w(x_0, y_0)$ is unique, so the Laurent series and symmetric sum of harmonics singular at the grid points differ only by a constant which comes from the outer integral.

APPENDIX 2

The method of singularities for transverse flow through a square array of cylinders.

Using the same grid as for parallel flow and the Stokes equations as in section 7 the boundary conditions on the simply connected quarter cell $ABCDE$ in figure 6 for flow in the x direction driven by a pressure gradient, $-P$, are as follows:

$$\begin{array}{llllll}
\text{On } AB & p = p_0 - Pl/2, & \partial u/\partial x = 0, & v = 0, & \partial\chi/\partial x = 0, & \partial\nabla^2\chi/\partial x = 0 \\
\text{On } BC & \dfrac{\partial p}{\partial y} = 0, & \partial u/\partial y = 0, & v = 0, & \chi = lU, & \nabla^2\chi = 0 \\
\text{On } CD & p = p_0, & \partial u/\partial x = 0, & v = 0, & \partial\chi/\partial x = 0, & \partial\nabla^2\chi/\partial x = 0 \\
\text{On } DE, & & u = 0, & v = 0, & \chi = 0, & \partial\chi/\partial r = 0 \\
\text{On } EA & \dfrac{\partial p}{\partial y} = 0, & \partial u/\partial y = 0, & v = 0, & \chi = 0, & \nabla^2\chi = 0
\end{array}$$

where U is the average velocity. The equations are Laplace and Poisson equations with no boundary condition for p on DE and reduce to a biharmonic equation for χ with mixed double boundary conditions on all sections of the boundary.

Given χ on CD and the existence theorem for the biharmonic equation or for Stokes flow as in Ladyzhenskaya (1969), χ is determined uniquely and so u, v and p may be found. Alternatively, if we are given P we can rescale U accordingly and find all the field quantities.

With square symmetry, grid periodicity and uniqueness of the solution, the unknown functions or their normal derivatives on the boundaries are expressible as unique Fourier series in x, y or θ_l though we do not yet know the coefficients in these series.

Using the Laplace and Poisson equations for p, u and v and the same Green's function as before we obtain Laurent series by integrating over the cell at the origin and second solutions which are sums of harmonics and biharmonics with identical singularities at each grid point plus a boundary integral over the rectangle, S_R.

Because of uniqueness the boundary integrals cancel any shape dependent terms in the sums and in particular there are no terms with odd symmetry if the grid is symmetrical ($I = J$ and $K = L$) and the grid sums converge if the pressure and shear forces balance. There remain two shape dependent constants P_2 and Q_4. The contribution to these from the outer boundary integral is zero if $K \gg I$. Otherwise we may determine these when we equate the two solutions by using the conditions that there are saddle points and maxima at A, B and C in (figure 6).

NOTATION

a	cylinder radius
b	distance between adjacent cylinder centres
b_l, α_l	polar coordinates of the cylinder centres
b_{pq}, α_{pq}	polar coordinates of the cylinder centres
c	density of cylinders, solid fraction
C_q	$\cosh^2 \pi\sqrt{3}q/2$
d_{pq}	complex coordinate of a grid point
F, F_1	force per unit length on a cylinder
l	distance between adjacent cylinder centres
p, q	integers
p	pressure
$-P$	pressure gradient in the x direction
P_n, Q_n	defined grid constants listed in Tables 1–4
Q	flux of fluid through a cell
r, θ	polar coordinates from the origin
r_{pq}, θ_{pq}	polar coordinates from (pl, ql)
S_q	$\sinh^2 q\pi$ for the square grid or $\sinh^2 \pi\sqrt{3}q/2$ for the triangular grid
U	mean fluid velocity (flux/total area of a cell)
u, v	x and y components of velocity
$\mathbf{v}$	vector velocity
v_r, v_θ	polar components of velocity
$w, w(r, \theta)$	velocity parallel to the z-axis
z, z_{pq}	complex coordinates
α_l, α_{pq}	polar angles of cylinder centres
ϵ	density of cylinders, solid fraction (1-void fraction)
θ, θ_{pq}	azimuthal angles
μ	viscosity
ϕ	Banerjee's velocity potential
χ	stream function